

# On the Need for Phenomenological Theory of $P$ -Vortices

## or

## Does Spaghetti Confinement Pattern Admit

## Condensed-Matter Analogies?

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Usually the intuition from condensed-matter physics is used to provide ideas for possible confinement mechanisms in gauge theories. Today, with a clear but puzzling “spaghetti” confinement pattern, arising after a decade of lattice computer experiments, which implies formation of a fluctuating net of peculiar magnetic vortices rather than condensation of the homogeneously distributed magnetic monopoles, the time is coming to reverse the logic and search for similar patterns in condensed matter systems. The main thing to look for in a condensed matter setup is the simultaneous existence of narrow tubes ( $P$ -vortices or 1-branes) of direction-changing electric field and broader tubes (Abrikosov lines) of magnetic field, a pattern dual to the one, presumably underlying confinement in gluodynamics. As a possible place for this search we suggest systems with coexisting charge-density waves and superconductivity.

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# 1 Introduction

A possible resolution of the confinement problem [1]-[12] should answer questions at two related but somewhat different levels:<sup>1</sup>

(i) It should allow for a reliable evaluation of various quantities, such as the gap in the spectrum of perturbations around the *true vacuum*, the string tensions in the area laws for the Wilson loops in different representations, as well as the masses of glueballs and other hadrons (when light quarks are taken into consideration).

(ii) It should provide a simple qualitative “picture” of how the vacuum is formed, how does the linear potential arise between remote sources with non-vanishing  $N$ -alities in the absence of light quarks and how are the massive colorless hadrons formed in the absence as well as in the presence of light quarks.

Of principal importance for development of theoretical (not computer-experimental) quantitative methods at level (i) would be identification of the true vacuum  $|vac\rangle$  – a functional of fields at a given moment of time, which is the lowest eigenstate of the non-perturbative Yang-Mills Hamiltonian, – with all the other eigenstates presumably separated from  $|vac\rangle$  by a non-vanishing gap.

The relevant approach to (ii) would rather identify a relatively small subspace in the space of all field configurations (labeled by a sort of collective coordinates) and substitute the original problem of Yang-Mills dynamics by that of a more or less familiar *medium* – QCD aether (like a gas of monopoles or  $P$ -vortices, a dual superconductor or something else). The underlying belief here is that the original functional integral at low energies gets dominant contribution from a restricted set of field configurations, and thus can be substituted by some more familiar effective theory, describing – at least qualitatively – the low-energy quantities as averages over this auxiliary *medium* and expressing the problems of low-energy quantum Yang-Mills theory through those of the *medium* dynamics.

*Understanding* of confinement requires certain achievements at both levels (i) and (ii): the existence of a “picture” is the thing that distinguishes “understanding” from just “calculability”, while the possibility to make calculations or at least estimates is a criterium for selection of a correct “picture” among the alternative ones.

The problem of confinement consists of two parts: one should explain, why

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<sup>1</sup>We discuss confinement as a pure gluodynamical problem and ignore all issues related to fermion condensates and chiral symmetry breaking. In the real-world QCD, the effects related to light quarks, can be more important for a large part of hadron physics and even the *dominant* confinement mechanism may be different [9]. Because of this, in the study of confinement in *gluodynamics* one should rely more upon computer than accelerator experiments.

We also do not dwell upon the promising “holistic” approaches to confinement, exploiting various general properties of gluodynamics [10] or building one or another kind of self-consistent approximation to correlation functions [11], [12]. Instead we discuss the lattice-experiment results, providing a *microscopic* description of relevant field configurations and their common properties and wonder if this mysterious pattern was ever observed in other types of physical systems.

( $\alpha$ ) all gauge fields are screened (i.e. all gluons, electric and magnetic, acquire effective masses  $\sim \Lambda_{QCD}$ ) and

( $\beta$ ) still, there is a peculiar *long-range* color-electric interaction, described by a narrow tube, where electric force lines (carrying a flux with non-vanishing  $N$ -ality, i.e. in representation, which can not be obtained in a product of adjoints, so that the tube is stable against string-breaking, caused by creation of a set of gluons) are collimated and give rise to the linear interaction potential  $V(R) \sim \sigma R$  at  $R \gg \Lambda_{QCD}^{-1}$ , with the string tension  $\sigma \sim \Lambda_{QCD}^2$  and the string width  $r_e \sim \Lambda_{QCD}^{-1} \log(R\Lambda_{QCD})$ .

We call this double-face situation the dual Meissner-Abrikosov (MA) effect.

The spaghetti vacuum pattern [6], to be discussed below, implies that in addition to ( $\alpha$ ) and ( $\beta$ ),

( $\gamma$ ) one more *long-range* interaction survives, described by a *very* narrow tube ( $P$ -vortex or 1-brane), with collimated color-magnetic force lines, populated by 0-branes, looking in certain aspects like magnetic monopoles and antimonopoles, with the direction of the magnetic field reversed at the locations of the 0-branes,

( $\delta$ ) the  $P$ -vortices can merge and split, they form a dense net, percolating through the whole volume.

Thus in some sense the dual MA effect is complemented by a kind of ordinary MA effect, though magnetic Abrikosov tubes carry a good deal of additional structure (moreover, as we will discuss below, the oversimplified description of this structure, as given in ( $\gamma$ ) is not gauge-invariant and thus is not fully adequate).

## 2 Screening in Abelian theory

It is well known that the MA effect *per se* does not require a non-Abelian gauge theory for its manifestation. It can be discussed already at the Abelian level.

There are many ways to obtain one or another kind of the screening effect ( $\alpha$ ) and many of them allow for one or another kind of long-range interactions to survive.

**Massive photon.** Complete screening with no long-range interactions is described by the effective Lagrangian of the type

$$\frac{1}{e^2} F_{\mu\nu}^2 + m^2 A_\mu^2. \quad (1)$$

It explicitly breaks gauge invariance and contains non-propagating degrees of freedom  $A_0$ , giving rise to instantaneous, but still screened, interaction.

**Debye screening.** It appears in ordinary conductors, electrolytes and some phases of plasma and is described by the effective Lagrangian

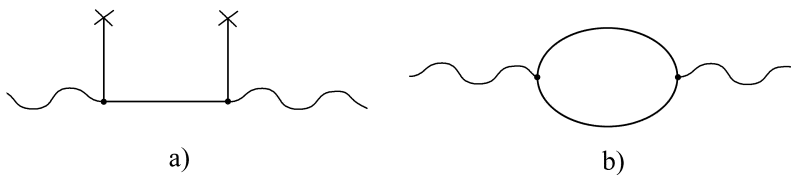
$$\frac{1}{e^2} F_{\mu\nu}^2 - E_i \frac{m^2}{\vec{\partial}^2} E_i. \quad (2)$$

It explicitly breaks Lorentz invariance and completely screens static electric fields, while magnetic and time-oscillating electric fields remain long-range. The massive term is usually produced by the process shown in Figure 1 and  $m^2$  is proportional to the concentration  $n_0$  of electric charges in the medium. If these charges are not originally present, then  $m^2 \sim n_0$  is either due to non-vanishing temperature, or, if the temperature is zero, to the probability of charge-anticharge creation by an imposed external electric field. This probability and thus  $m^2$  normally contains extra powers of 4-momenta, so that the screening mechanism gets essentially softened and leads, for example, to the slow **running coupling** phenomenon in QED and QCD, described (in these Lorentz-invariant cases) by the effective Lagrangian

$$F_{\mu\nu} \frac{1}{e^2(\Delta)} F_{\mu\nu}. \quad (3)$$

In  $3 + 1$  dimensions the  $\Delta$ -dependence is just logarithmic, at least in the leading approximation, so that no real screening takes place, gauge fields remain massless. In non-Abelian theories magnetic interactions also enter the game, producing the anti-screening effect in (3), overweighting the screening one [13]. It is not quite clear whether just this anti-screening could lead to the confinement effect when moving beyond the leading logarithm approximation (see, e.g., [11]).

**Figure 1.** The origin of the gauge-field mass in Debye-screening mechanism:



a): The case when charged particles are originally present in the medium. The entire diagram is proportional to the concentration  $n_0$  of these particles in the medium. For non-vanishing temperatures (unavoidable in any lattice calculations)  $n_0$  is never zero (but can be exponentially small).

b): The case when the charged pairs are created in the medium (including the physical vacuum) by the gauge field itself. In this case the screening is usually much softer and can result in a slow running of a coupling constant rather than in exponential screening.

To be more precise, in realistic systems the effective Lagrangian (in the case of linear response, i.e. weak fields) is expressed in terms of the dielectric constant <sup>2</sup>  $\epsilon_{ij} \equiv \left( \delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) \epsilon_{\perp}(\omega, \vec{k}) + \frac{k_i k_j}{\vec{k}^2} \epsilon_{\parallel}(\omega, \vec{k})$ :

$$\mathcal{L} = \frac{1}{e^2} \left[ F_{\mu\nu}^2 + (\epsilon_{\perp} - 1) \vec{E}^2 + (\epsilon_{\perp} + \epsilon_{\parallel}) \left( \text{div} \vec{E} \frac{1}{\vec{\partial}^2} \text{div} \vec{E} \right) \right] \quad (4)$$

and is not universal, since the frequency and momentum dependence of  $\epsilon_{\parallel}$  and  $\epsilon_{\perp}$  can be very different in different regimes. Important for the Debye screening (long-distance exponential decay of the field correlator) is the presence of a singularity in the longitudinal dielectric constant at large distances (small  $\vec{k}^2$ ) [15]:  $\epsilon_{\parallel} = 1 + \frac{\epsilon^2 m^2}{\vec{k}^2} + \mathcal{O}(\omega)$ , where the omitted terms describe highly non-trivial frequency dependence. Indeed, the static correlator is

$$\langle E_i E_j \rangle \sim \frac{k_i k_j}{\epsilon_{\parallel} \vec{k}^2} = \frac{k_i k_j}{\vec{k}^2 + P_{00}} \quad (5)$$

where  $P_{00} = (\epsilon_{\parallel} - 1) \vec{k}^2$  is the static value of the component of the photon polarization operator  $P_{\mu\nu}$  ("electric" mass [16]).

**Dual Debye screening.** It would be described by a dual effective Lagrangian of the type

$$\frac{1}{e^2} F_{\mu\nu}^2 + H_i \frac{m^2}{\vec{\partial}^2} H_i \quad (6)$$

and imply screening of static magnetic fields. It is unclear if any condensed-matter systems with this type of behaviour have been already discovered. In ordinary electrodynamics without magnetic charges, we have a counterpart of (5):

$$\langle H_i H_j \rangle = \frac{\vec{k}^2 \delta_{ij} - k_i k_j}{\vec{k}^2 + P} \quad (7)$$

where the "magnetic" mass  $P$  is given by the static value of the spatial components of the photon polarization operator ( $P_{ij} \stackrel{\omega=0}{=} \left( \delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) P$  due to the gauge invariance). In a gas of magnetic monopoles it becomes (see Polyakov's book in [1])

$$\langle H_i H_j \rangle = \delta_{ij} - \frac{k_i k_j}{k^2 + M^2} \quad (8)$$

**Chern-Simons screening.** It is described by the peculiar gauge invariant Lagrangian,

$$\frac{1}{e^2} F_{\mu\nu}^2 + m^{\alpha\ldots\beta} \epsilon_{\mu\nu\lambda\alpha\ldots\beta} A_{\lambda} F_{\mu\nu}. \quad (9)$$

It describes aspects of the Hall effect and related phenomena, is Lorentz invariant ( $m$  is a scalar) only in  $2+1$  dimensions, and – only in this dimension – makes the photon massive, but still the *long-range* Aharonov-Bohm interaction survives [17].

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<sup>2</sup>Note that the formulation in terms of dielectric constant and magnetic permeability  $\mu$  can be useful in the search for solid-state counterparts of the confinement phenomenon (see e.g. [14]): the electric confinement (like that in QCD) can be described by  $\epsilon = 0$ , while the magnetic confinement (like the Meissner effect in superconductors) is attributed to  $\mu = 0$ .

**Abelian Higgs model.** The ordinary (not the dual) Meissner-Abrikosov effect is modelled by the Abelian Higgs (Landau-Ginzburg) effective Lagrangian

$$\frac{1}{e^2} F_{\mu\nu}^2 + |D_\mu \phi|^2 + \lambda(|\phi|^2 - m^2)^2. \quad (10)$$

After  $\phi$  condenses,  $\langle \phi \rangle = m e^{i\theta}$ , the gauge fields become massive, thus giving rise to effect ( $\alpha$ ): the Meissner effect for magnetic and electric fields. However, actually the mass is acquired not by  $A_\mu$  field, but rather by a gauge invariant combination  $\hat{A}_\mu = A_\mu - \partial_\mu \theta$ , thus the mode  $\hat{A}_\mu = 0$  can still propagate through large distances, and this explains the effect ( $\beta$ ): emergence of Abrikosov tubes.  $\hat{A}_\mu = 0$  does not imply that  $A_\mu = \partial_\mu \theta$  is pure gauge, if  $\theta(x)$  is singular and  $\oint_C A_\mu dx^\mu \neq 0$  for some contours  $C$ . In an Abrikosov tube stretched along the  $z$  axis  $\theta = \arctan \frac{y}{x}$  is the angle in the  $xy$  plane and  $C$  is any contour in this plane, encircling the origin. Since  $\theta$  is the phase of the smooth field  $\phi$ , the modulus  $|\phi|$  should vanish on the  $z$  axis, where  $\theta$  is not well defined, i.e. the condition  $|\langle \phi \rangle| = m$  is destroyed in the vicinity of  $z$  axis, in a tube with the cross-section  $\Sigma = \pi r_m^2$ . This causes the energy  $\lambda m^4 \Sigma$  per unit length of the tube, while the energy of magnetic flux  $\Phi$  in the tube is  $\sim \left(\frac{\Phi}{\Sigma}\right)^2 \Sigma = \frac{\Phi^2}{\Sigma}$ . Minimization of the sum of these terms with respect to  $\Sigma$  defines the characteristic width of the tube  $\Sigma_m = \pi r_m^2 \sim \frac{\Phi}{\sqrt{\lambda} a^2}$ .

If electric charges  $q$  smaller than that of the Higgs field  $\phi$  are present in the theory, then  $q\Phi$  can be smaller than 1 and Aharonov-Bohm effect will be observed when such charges travel around the Abrikosov tube at any distance: thus, even though all gauge fields are massive, the Aharonov-Bohm interaction also remains long-range (unscreened) [18].

The technical reason allowing *magnetic* Abrikosov lines to exist is that the equation  $F_{xy} = \delta(x)\delta(y)$  can be easily resolved:  $A_x = \partial_x \arctan \frac{y}{x}$ ,  $A_y = \partial_y \arctan \frac{y}{x}$  and the Higgs field just provides a source of the needed form, with *electric* current  $J_x = \partial_y F_{xy} = \delta(x)\delta'(y)$ ,  $J_y = -\partial_x F_{xy} = -\delta'(x)\delta(y)$  rotating around the  $z$ -axis.

In order to obtain an *electric* Abrikosov line one would need to solve the equation  $F_{0z} = \delta(x)\delta(y)$ , which violates Bianchi identity and requires the existence of a *magnetic* current (rotating around the  $z$ -axis) and thus, in a Lorentz invariant setting, of magnetic charges (monopoles).<sup>3</sup> Thus, in order to describe confinement with properties ( $\alpha$ ) and ( $\beta$ ), where the *dual* MA effect is needed, one often makes use of the **dual Abelian Higgs model** (the dual superconductor model), where the Higgs field  $\tilde{\phi}$  is magnetically charged, i.e. interacts with the dual field  $\tilde{A}_\mu$ , such that  $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} = \epsilon_{\mu\nu\alpha\beta} \partial_\alpha A_\beta$ . In this type of scenarios the role of non-Abelian degrees of freedom is thought to be the imitation of Higgs degrees of freedom (see, for example,  $W^\pm$  in eq.(20) below and ref.[19]) and the problem is to find a mechanism, leading to their appropriate condensation.

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<sup>3</sup>Similarly, in order to have a *magnetic* tube, where the field is not constant along the line, in particular it changes direction at some points  $z_a$ , one needs to solve an equation  $F_{xy} = \frac{1}{2} \delta(x)\delta(y) \prod_a \text{sign}(z - z_a)$  which violates Bianchi identity at  $x = y = 0$ ,  $z = z_a$  and thus requires magnetic charges (monopoles) at these points.

As already mentioned, the lattice experiments (see Section 4 below) imply that the real pattern (and, perhaps, the mechanism) of confinement can be more sophisticated and may imply coexistence of  $(\beta)$  electric and  $(\gamma)$  structured magnetic tubes. Therefore, it is important to note that no Abelian model is known, which would allow the coexistence of magnetic and electric MA effects, e.g. no effective Lagrangian of the form

$$\frac{1}{e^2} F_{\mu\nu}^2 + m_m^2 \hat{A}_\mu^2 + m_e^2 \hat{\tilde{A}}_\mu^2 \quad (11)$$

is allowed. Therefore, if such coexistence is not an artifact of lattice experiments (what is not considered too probable nowadays), it requires construction of more sophisticated models. A natural hope is that such models can be straightforwardly built in modern string theory (involving branes) and realized in condensed-matter systems.

Note that what is needed is some kind of restoration, at least partial, of electro-magnetic duality, present in Abelian photodynamics. This duality is usually broken by all known relevant modifications: by the introduction of electric charges (without adding their magnetic counterparts), by embedding into non-Abelian theory (where electric and magnetic interactions of gluons are different), by the addition of a Chern-Simons term, or by coupling to Higgs scalars and going to a superconducting phase. Lattice experiments strongly suggest the need for some – yet unstudied – (topological, i.e. with the field-content of field, not string, theory) stringy phases with both “fundamental” and  $D1$  strings present, where screening and MA phenomena do not contradict electromagnetic duality.

### 3 $3d$ compact QED

The sample example [2] of confinement proof in Abelian  $2 + 1$  dimensional compact electrodynamics (embedded into the non-Abelian Georgi-Glashow model to justify compactness and provide ultraviolet regularization, rendering finite the instanton action), deals actually with *random confinement* [6, 12] and with *Wilson’s confinement criterion* [1]: no fluxes acquire vacuum averages, only their squares,  $\langle \Phi \rangle = 0$ ,  $\langle \Phi^2 \rangle \neq 0$ , and this is enough to provide the area-law behaviour for the Wilson-loop averages. In this example the relevant *medium* in 2 space dimensions is obtained as a time-slice of an instanton gas with Debye screening. Instantons in Abelian  $2 + 1$ -dimensional theory are just ordinary 3-dimensional monopoles and antimonopoles with magnetic fields

$$H_\mu = \epsilon_{\mu\nu\lambda} F^{\nu\lambda} = \pm g \frac{r_\mu}{r^3}, \quad (12)$$

or rather

$$H_\mu = \pm g \frac{r_\mu}{(r^2 + \varepsilon^2)} e^{-r/\xi}, \quad (13)$$

where  $\epsilon$  and  $\xi$  provide ultraviolet (from the underlying non-Abelian theory) and infrared (from the Debye screening in the monopole-antimonopole gas) regularizations respectively;  $g$  is the monopole

charge, normalized so that  $2eg = \text{integer}$ . Thus the *medium* looks like a set of appearing and disappearing vortex-antivortex pairs with the pseudoscalar  $2d$  magnetic and vector  $2d$  electric fields

$$\begin{aligned} B &= \epsilon_{ij} F^{ij} = \pm \frac{gt}{(\vec{x}^2 + t^2)^{3/2}}, \\ E_i &= F_{0i} = \pm g \frac{\epsilon_{ij} x^j}{(\vec{x}^2 + t^2)^{3/2}}. \end{aligned} \tag{14}$$

The field  $E_i$ , produced by the time-variation of  $B$ , has non-trivial vorticity and thus contributes to the rectangular Wilson average over this medium

$$\langle \exp \left( ie \oint_C (A_0 dt + A_i dx^i) \right) \rangle = \langle \exp \left( ie \int_S \vec{E} \cdot d\vec{x} dt \right) \rangle \tag{15}$$

where the contour  $C$  lies in the  $xt$  plane and  $S$  intersects the  $xy$  plane by a segment  $\tilde{C}$ . The contribution of a vortex to the integral  $\int_{\tilde{C}} \vec{E} \cdot d\vec{x}$  is equal to

$$\pm \int_{-L}^L \frac{y dx}{(x^2 + y^2 + t^2)^{3/2}} \sim \pm \frac{2y}{y^2 + t^2} \tag{16}$$

for  $L \gg \sqrt{y^2 + t^2}$  (with the distance  $\sqrt{y^2 + t^2}$  actually bounded from above by the Debye radius  $\xi$ ) and further integration over  $t$  gives for a contribution of a vortex:

$$\pm 4\pi g \Phi = \pm 2\pi g \frac{y}{|y|} = \pm 2\pi g \text{sign}(y), \tag{17}$$

provided the vortex lies in a slice of width  $\xi \ll L$  around the surface  $S$ . This flux is one-half of the full flux  $4\pi g$  of the charge- $g$  monopole. The factor  $1/2$  appears here because only half of the vorticity of  $\vec{E}$  contributes to the integral. Since contributions of vortices and antivortices have opposite signs, the average of  $\int \vec{E} d\vec{x} dt$  itself is of course vanishing, but the even powers of this integral, and thus the Wilson exponent, can have non-vanishing averages. The simplest estimate with the help of Poisson distributions gives [20]:

$$\begin{aligned} &\langle \exp \left( ie \oint_C (A_0 dt + A_i dx^i) \right) \rangle = \\ &= \sum_{n_+, n_- = 0}^{\infty} \left[ e^{-\bar{n}} \frac{\bar{n}^{n_+}}{n_+!} \right] \left[ e^{-\bar{n}} \frac{\bar{n}^{n_-}}{n_-!} \right] e^{4\pi i e g (n_+ - n_-) \Phi} = e^{-2\bar{n}(1 - \cos(4\pi e g \Phi))} \end{aligned} \tag{18}$$

Since the average number of contributing vortices and antivortices is  $\bar{n} = \xi A_S n_0$ , where  $A_S$  is the area of the surface  $S$  and  $n_0$  is the concentration of vortices (depending, primarily on the instanton action, which in turn is defined by the ultraviolet regularization), one obtains the area law for the Wilson loop, at least for the minimal value  $eg = 1/2$  allowed by the Dirac quantization condition<sup>4</sup>. Similarly, one could calculate the average

$$\langle \exp \left( ie \oint_C (A_0 dt + A_i dx^i) \right) \rangle = \langle \exp \left( ie \int_S B dx dy \right) \rangle \tag{19}$$

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<sup>4</sup>There are corrections to this oversimplified calculation [21, 20], which in particular can destroy the prediction of (18), that confinement disappears for *even* magnetic charges (when the relevant flux  $\Phi$  is integer).



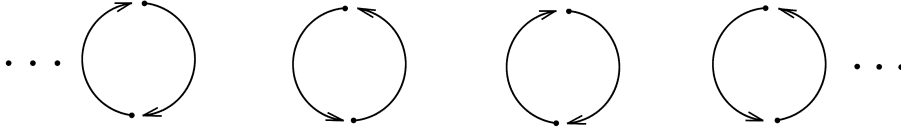
of the space-like Wilson loop, with  $S$  lying in the  $xy$ -plane and bounded by the curve  $C$ . This average is given by the same formula (18).

Another *interpretation* of the same calculation [3], implies that the distribution of vortices is *affected* by the presence of the loop, so that the vortices and antivortices get concentrated around the surface  $S$  and *screen* it.

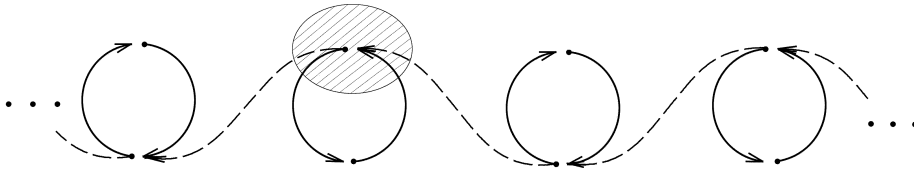
## 4 Confinement in $4d$

In  $3 + 1$  dimensions no such simple calculation from first principles is yet known. The main difference is that ordinary instantons in  $3 + 1$  dimensions are no longer charged: their field vanishes too fast at infinity and, therefore, the confinement mechanism should involve an additional *dissociation* of instantons into something like magnetically charged *merons* [3, 22]. Time slices of instantons are now  $3d$  objects, namely monopole-antimonopole pairs (if looked at in a special gauge), and the instanton describes the process of their spontaneous creation and annihilation.

**Figure 2.** Possible phases of the recombinant plasma of instanton gas.

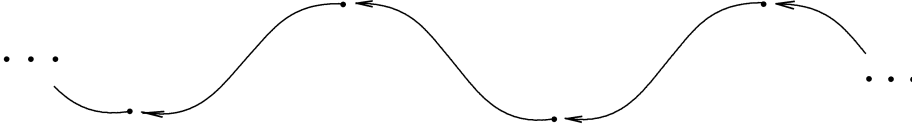


a): *Recombinant phase* (ordinary instanton gas in  $3 + 1$  dimensions): each instanton is the process of creation and annihilation of a monopole-antimonopole pair.



b): Transition to *jumping recombinant phase* (instantons dissociate into merons): created pairs do not coincide with annihilating pairs. The shadowed domain corresponds to the meron.

What one expects is that in the dense instanton gas (or liquid) recombination takes place between monopoles and antimonopoles from *different* pairs, thus *picking up* a *chain* of instantons from the



c): *Jumping recombinant phase*: a *chain* is naturally formed.

liquid (see Figure 2).

The spaghetti vacuum pattern implies that such chains are actually spread out through the entire volume and form a “percolating cluster” [20, 23].

As in the 2 + 1-dimensional case, the electric fields with non-vanishing vorticities, caused by the moving monopoles and antimonopoles, contribute to the Wilson averages in 3 + 1 dimensions and give rise to the area laws.

At the moment there is no absolutely convincing *theoretical* argument in favour to this kind of ideas; instead they obtained considerable support from computer *experiments*.

## “Experimental” lattice results

Lattice computer simulations are primarily targeted at producing qualitative results in the spirit of (i) and thus at providing a proof that the Yang-Mills functional integral indeed describes a theory with a mass gap, a linear potential, a realistic hadronic spectrum and realistic hadron interactions. Remarkably enough, these experiments could also be used for research in direction (ii) and they indeed produced very inspiring results. However, up to now the simulations are not very detailed and one actually substitutes the functional integral by a sum over a rather small random subset of field configurations, which are believed to give the dominant contribution. According to (ii) one can hope that most of these dominant configurations will have something in common – and this is what really happens – providing a clear description of the *medium* required in (ii).

This *experimentally* discovered [24, 25] medium appears to be somewhat unexpected (see [6] for the original suggestion of this “Copenhagen spaghetti vacuum” and [26] for comprehensive modern reviews and references): it turns out to be filled with peculiar *one-dimensional* objects (with two-dimensional world surfaces) – *P*-vortices – which in a certain Abelian approximation (see next subsection) look like narrow (of width  $r_m \ll \Lambda_{QCD}^{-1}$ ) tubes of *magnetic* field, directed along the tube and changing direction to the opposite at locations of monopoles and antimonopoles, which form a 1-dimensional gas inside the tube <sup>5</sup>. Such objects are obviously stable against the creation of monopole-antimonopole pairs:

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<sup>5</sup>In contrast to the *P*-vortices themselves, the monopoles and antimonopoles inside them are difficult to define in a

such processes can not break the tube into two, because the magnetic flux through any section outside the monopole cores is  $1/2$  <sup>6</sup>. The net of these direction-changing color-magnetic tubes fills the entire space [20] (forming a “percolating cluster” [23]) <sup>7</sup>, and in this medium the force lines of color-electric fields (emitted by sources of non-vanishing  $N$ -ality) also form tubes (of width  $r_e \sim \Lambda_{QCD}^{-1}$ ), thus giving rise to the confinement phenomenon. In lattice *experiments* the area laws for appropriate Wilson-loop averages are explicitly checked and the  $P$ -vortices from percolating cluster are shown to give dominant contribution to the string tensions. *Theoretically*, the contribution of  $P$ -vortices to the string tension depends on their abundance and one of the tasks of the theory is to explain the origin of the medium of  $P$ -vortices and how it is consistent with Lorentz invariance.

So far there is no clear theoretical explanation of why and how such a medium is formed in non-Abelian gauge theories and why – once formed – it can give rise to a dual Meissner effect and lead to confinement, though the (lattice) experimental evidence in favour of this pattern is rapidly growing.

A serious drawback of the published results of lattice experiments is that they do not provide the essential information about instanton-like and meron-like configurations and their probable association with the *localized*  $P$ -vortex clusters and, furthermore, they do not explicitly study the configurations of collimated color-electric force lines between sources with non-vanishing  $N$ -ality (which do not need to be fermions). Information about these color-electric tubes is extracted indirectly from the study of Wilson averages. This is not enough to understand what happens to these tubes, say, after the maximal Abelian projection, and whether their content indeed looks like Abelian electric field exactly in the same projection, when the  $P$ -vortices look like the tubes of direction-changing Abelian magnetic field. Any data touching upon this issue would be extremely useful for further clarification of the situation.

## Maximal Abelian Projection

The “ $P$ ” in “ $P$ -vortices” comes from the word “projection” [29]. It is inspired from the way they are often searched for and studied, which is not gauge invariant, even though the  $P$ -vortices themselves are in fact gauge invariant (see Figure 3).

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gauge-invariant way. Even the direction of the would be Abelian magnetic field and thus the exact positions of monopoles and antimonopoles inside the  $P$ -vortex are unphysical: they can be changed by gauge transformations. Indeed, to change the *direction* of an Abelian field strength  $F_{\mu\nu}^3$  at a given point it is enough to make a singular gauge transformation, conjugating the fields by a unitary matrix like  $\sigma^1$  at this point (though it is not absolutely clear how to make such operation consistent with the maximal Abelian projection, described in the next subsection). There is still controversy in the literature (see, for example, [27] for different points of view) about the actual internal structure of the  $P$ -vortices and the (dis)advantages of visualizing it in terms of monopoles and antimonopoles.

<sup>6</sup>This does not contradict the possibility that isolated monopoles are screened [28].

<sup>7</sup>In addition to the precolating cluster, there exists also a variety of non-percolating ones, also populated by monopoles. There is no agreement in the literature on whether these non-percolating clusters are lattice UV-artifacts or they actually contribute in the continuum limit.

Usually, one uses a procedure called the Maximal Abelian Projection (MAP) <sup>8</sup>. It splits into two steps. First, for every configuration of the fields  $A_\mu^a(x)$ , taken with the weight dictated by the true non-Abelian action, the “maximal Abelian gauge” is chosen, by minimizing the lattice counterpart of  $\int W_\mu^+ W_\mu^-(x) d^4x$  along the gauge orbit. This first step is absolutely justified (though technically it suffers from ambiguities caused by the existence of Gribov copies).

**Figure 3.** This figure, borrowed from the seminal paper [25], is the best existing illustration of what  $P$ -vortices are and what the maximal Abelian projection does.

This allows one to introduce the induced effective action  $\tilde{S}(A)$ , obtained after integration over the other components ( $W_\mu^\pm \equiv A_\mu^1 \pm iA_\mu^2$ ,  $D_\mu(A) \equiv \partial_\mu + ieA_\mu^3$ )

$$\begin{aligned} \exp(-\tilde{S}(A)) &= \int DW^+ DW^- \delta\left(\left|D_\mu(A)W_\mu^+\right|^2\right) \det_{FP}^2(\partial_\mu D_\mu(A)) \times \\ &\times \exp\left[-\frac{1}{g^2}\left[\left(F_{\mu\nu} + (W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+)\right)^2 + |D_\mu(A)W_\nu^+|^2\right]\right] \end{aligned} \quad (20)$$

At the second step, one makes use of  $\tilde{S}(A)$  to define Abelian correlation functions

$$\langle \prod_i \mathcal{O}(A_\mu^a) \rangle_{MAP} \equiv \langle \prod_i \mathcal{O}(W_\mu^\pm = 0, A_\mu^3) \rangle = \int DA_\mu^3 e^{-\tilde{S}(A_\mu^3)} \prod_i \mathcal{O}(W_\mu^\pm = 0, A_\mu^3) \quad (21)$$

This step implies that one makes use of the true non-Abelian action, i.e. includes contributions from the *virtual*  $W^\pm$ -bosons in loops, but omit them from external lines. Therefore, the second step – the *projection* itself – is an approximation:

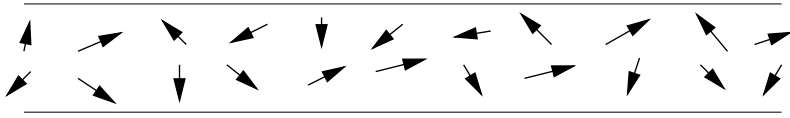
$$\langle \prod_i \mathcal{O}(W_\mu^\pm, A_\mu^3) \rangle \neq \int DA_\mu^3 e^{-\tilde{S}(A_\mu^3)} \prod_i \mathcal{O}(W_\mu^\pm = 0, A_\mu^3) \quad (22)$$

Its experimentally discovered [31] surprising efficiency (as compared with the complete answer including non-Abelian fields) is often called the *hypothesis of Abelian dominance*. Though theoretically so far unjustified and uncontrollable, it provides a convenient *language* for description (visualization) of the confinement phase: it is at this level that monopoles and antimonopoles appear. Figure 3 can serve as an illustration of how the MAP works.

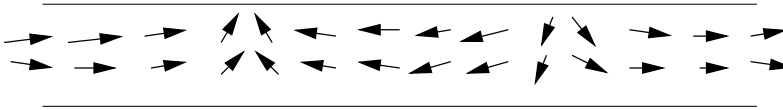
The theoretical problem of evaluation of  $\tilde{S}(A)$  remains open. See [19] for interesting attempts to identify condensating modes and vortex-like structures in the functional integral (20) and [32] for a supersymmetric model with BPS configurations, which look like magnetic  $P$ -vortices populated by monopoles.

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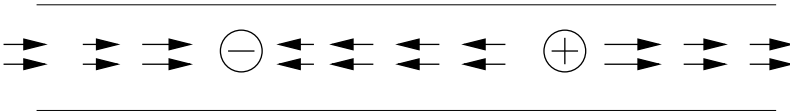
<sup>8</sup>Comparison with the results of lattice experiments in other Abelian approximations usually demonstrates that the (gauge non-invariant and necessarily approximate) language of monopoles is most reliable in the MAP, the use of this language in other calculational schemes can often be misleading [30].



a): A fragment of the distribution of field strength in an original configuration of fields  $A_\mu^a(x)$ , from the set of those which give dominant contribution to the non-Abelian functional integral. The strenghts are non-vanishing within a narrow tube, the  $P$ -vortex. Actually, the entire configuration looks like a net of  $P$ -vortices, containing the “percolating cluster”, which has proper scaling properties and survives in the continuum limit. The arrows indicate directions in *color* space.



b): The maximal Abelian gauge is chosen, which minimizes  $\int W_\mu^+ W_\mu^-(x) d^4x$ . It is just a choice of gauge (field strenghts are rotated), no approximation is involved. Certain structures are clearly seen in the distribution of field strenghts *inside* the tube.



c): Maximal Abelian *projection* is performed:  $W_\mu^\pm(x)$  are put equal to zero. The *structures* seen in (b), turn into a clear (but approximate) pattern of collimated magnetic force lines, changing direction at the location of monopoles and antimonopoles. No peaks of magnetic energy occur at these locations.

## 5 Are there condensed-matter analogies of confinement?

Coming back to the lattice results above, a natural question to ask is if anything similar can be found in other avatars of gauge theories, for example, in condensed matter or plasma physics. There, one would rather expect to encounter a dual type of medium: electric  $P$ -vortices, formed by chains of positive and negative electric charges, connected by narrow tubes of electric fields with fluxes  $\pm 1/2$ , *and* an ordinary (magnetic) MA effect, implying formation of magnetic-field tubes with a constant unit flux (and confinement of hypothetical magnetic charges), *caused by* or at least *consistent with* the existence of such electric  $P$ -vortices. In condensed matter analogues, the underlying non-Abelian Yang-Mills dynamics responsible for the formation of  $P$ -vortices, should presumably be replaced by

some other dynamics (additional forces), allowed in condensed matter systems. The whole situation (the coexistence and even mutual influence of electric  $P$ -vortices and magnetic MA effect) is already exotic enough to make one wonder if anything like this can at all occur in any kind of natural matter systems.

The main thing to look for in a condensed matter setup is the **simultaneous existence of narrow tubes ( $P$ -vortices) of direction-changing electric field and broader tubes (Abrikosov lines) of magnetic field** – a dual pattern to the one, underlying the spaghetti confinement mechanism of gluodynamics. This clearly implies, that superconductivity (from the dual superconductor scenario), if relevant at all, should be of a more sophisticated nature than just the single-field condensation (monopole condensation), the superconducting order should be caused or at least coexist with an order of some other type (responsible for the formation of  $P$ -vortices). This looks almost like the requirement that the Meissner-Abrikosov effect (for magnetic field) *coexists* with (or, perhaps, is even *implied by*) the dual Meissner-Abrikosov effect (for electric field), but actually the tubes of electric field should be different: they should have internal structure, namely the one-dimensional gas of positive and negative electric charges, an electric field along the tube which changes direction at the locations of these charges and be stable against possible “string breaking”, caused by creation or annihilation of charge-hole pairs. Moreover, the width of electric tubes should/can be different (much smaller?) than that of magnetic tubes.

The main goal of this paper is to bring these issues to the attention of experts in other fields, such as condensed matter and plasma physics and to emphasize the fact that the discovery of a similar picture arising under any circumstances, would be of great help for the development of the confinement theory and in particular for the understanding of possible  $2d$  vortex theories, living on the world sheets of the relevant branes, as well as of the phase structure of these theories<sup>9</sup>. If, on the contrary, no such pattern exists in condensed matter physics, this would once again emphasize the peculiarities of non-Abelian gauge theories (where elementary quanta carry more structure than just point-like charges and thus the naive screening behaviour is from the very beginning substituted by antiscreening and further non-naive phenomena are naturally expected to occur).

The rest of this paper is purely speculative, added for encouragement: in order to demonstrate that superconductivity (probably responsible for the magnetic Meissner-Abrikosov effect) can indeed

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<sup>9</sup>Among other things, it would be interesting to exploit the idea of the *topological* confinement which, in different versions, often works in condensed matter physics. A characteristic feature of the topological confinement is that it depends on the dynamics of the theory only through the properties of particular excitations (quasiparticles), while their interactions do not matter. For example, one-dimensional objects can be tied and, therefore, be unseparable, and this can work for real one-dimensional excitations, like Abrikosov tubes, and for point-like magnetic monopoles and/or hedgehogs, which have attached Dirac strings. In practice, topological confinement can look very similar to the mechanism we discuss throughout the paper. See [33, 34] for some examples, see also [35].

*coexist* with at least some kind of *dual* order (though the example below falls short from exhibiting narrow tubes of direction-changing electric field).

## Charge density waves

As a possible (but by no means the only) candidate analogue of the electric  $P$ -vortices we would like to suggest the *charge density waves* (CDW) and the questions which arise are:

- (a) are there any *tube-like* CDW with a charge density similar to  $\rho(x, y, z) \sim \delta^{(2)}(x, y) \sin z$  and (perhaps, direction-changing) electric force lines collimated along the  $z$  axis?
- (b) can the CDW *coexist* with superconductivity (SC), which would be a natural reason for the Meissner-Abrikosov effect?
- (c) can the CDW *cause* or at least *enhance* superconductivity?
- (d) can the widths of the CDW-like  $P$ -vortices be much smaller than those of Abrikosov lines (where the Cooper-Higgs-like condensate is broken)?

Remarkably, a very similar set of questions is currently under intense investigation in connection with high- $T_c$  superconductivity (where the adequate theoretical pattern also remains unknown), and it looks like the above possibilities are indeed open, as one may see in [36] and references therein. Of course, the real media appearing in condensed matter examples, have a lot of additional structure (primarily, the highly anisotropic crystal lattice in the background, playing a key role in the formation of realistic CDW), which one does not expect to find in gluodynamics. For closer analogies with gluodynamics one can also look for phenomena in liquid  $He$  [33], dense relativistic plasma, segnetoelectrics [14] or even biological membranes [37]. Still, we want to emphasize once again that today, when the formulation of a *phenomenological theory* of  $P$ -vortices is so important, one needs to consider *all* examples, where objects of this kind are presumably present, irrespective of the underlying microscopic structure, and the solid-state systems with the coexisting CDW and SC orders should not be neglected – especially because, like the confinement in gauge theories, they are now under close scrutiny, and considerable progress can result rather fast from comparison of ideas from the two fields.

The simplest facts and ideas about the CDW-SC systems, though not immediately coinciding with (a)-(d), do not seem to be in obvious contradiction also. The relevant list of properties seem to include the following:

- The CDW formation causes transition to an insulator phase (Peierls-Frohlich-Mott transition), while SC transition gives rise to a (super)conductor.
- Thus CDW and SC orders compete with each other, with CDW usually stronger competitor than SC [38].
- Still the CDW and SC orders can coexist [39, 40].
- Even if both CDW and SC orders are not established simultaneously at long distances, they

interfere locally, one phase appears in the regions where the other is broken: SC appears in the vicinity of CDW vortices and CDW appear in the vicinity of Abrikosov lines [40]. This can be enough, for example, to get the SC phase in when CDW dislocations percolate through the entire volume.

The phenomenological description of the CDW is in terms of electron-phonon interactions [41]. Notice that the vector nature of phonons makes them closer to the  $W$ -fields in (20) than the scalar fields, employed in the Abelian Higgs model (10).

## 6 Conclusion

The *theory* of the *Copenhagen spaghetti vacuum* should, of course, be developed in the context of string theory. The appropriate name for  $P$ -vortices is 1-branes. Monopoles living on these 1-branes are, naturally, 0-branes. The coexistence of electric and magnetic Abrikosov tubes should be modelled by that of coexisting “fundamental strings” and  $D1$  branes. The problems, raised in this paper, are related to the lack of any “underlying model”, for which the theory of strings and branes would be an effective model, the lack which seriously undermines the progress in modern string theory. We emphasize that the spaghetti vacuum in gluodynamics can by itself provide such a model and we also suggest to start a more extensive search for possible underlying models in modern condensed-matter physics.

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